Appendix E: Sigma Notation

Dr Craig

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Introduction

Who:
Dr Craig

What:
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Important information

Course code: MAT01A1

NOT: MAT1A1E, MAT1A3E, MATE0A1, MAEB0A1, MAA00A1, MAT00A1, MAFT0A1

Learning Guide: available on Blackboard. Please check Blackboard twice a week.

Student email: check this email account twice per week or set up forwarding to an address that you check frequently.
Important information

Lecture times: Tuesday 08h50 – 10h25
Wednesdays 17h10 – 18h45

Lecture venues: C-LES 102, C-LES 103

Tutorials: Tuesday afternoons (only!)
13h50 – 15h25: D-LES 104 or D-LES 106
OR

15h30 – 17h05: C-LES 203 or D1 LAB 408
Important information

Textbook: the textbook for this module is

Calculus: Early Transcendentals
(International Metric Edition)
James Stewart
7th edition
Other announcements

- 2nd year IT students who have a clash with the tutorial on a Tuesday afternoon must email Dr Craig today.
- No MAT01A1 tuts on Wednesdays. If you see this on your timetable, it is an error.
- Need help? Visit the Maths Learning Centre in C-Ring 512:
  10h30 – 14h35 Mondays
  08h00 – 15h30 Tuesday to Thursday
  08h00 – 12h55 Fridays
Lecturers’ Consultation Hours

Monday:
10h30 – 11h30 Ms Richardson (C-503)

Wednesday:
14h30 – 16h00 Ms Richardson (C-503)

Thursday:
11h00 – 13h00 Dr Craig (C-508)
13h30 – 14h00 Ms Richardson (C-503)

Friday:
11h30 – 13h00 Dr Craig (C-508)
Sigma notation

If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and $m, n \in \mathbb{Z}$ such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

The letter $i$ is called the **index of summation**. Other letters can also be used as the index of summation.

The number of terms in the sum is $n - m + 1$. 
Examples:

(a) \[ \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 \]

(b) \[ \sum_{i=3}^{n} i = 3 + 4 + 5 + \ldots + (n - 1) + n \]

(c) \[ \sum_{j=0}^{3} 2^j = 2^0 + 2^1 + 2^2 + 2^3 = 15 \]

(d) \[ \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \]
More examples:

(e) \[ \sum_{i=1}^{3} \frac{i - 1}{i^2 + 3} = \frac{1 - 1}{1^2 + 3} + \frac{2 - 1}{2^2 + 3} + \frac{3 - 1}{3^2 + 3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42} \]

(f) \[ \sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8 \]
Exercise: Write the sum $2^3 + 3^3 + \ldots + n^3$ in sigma notation.

Solution(s): Sigma notation for a particular sum is not unique. Some possible solutions:

- $2^3 + 3^3 + \ldots + n^3 = \sum_{i=2}^{n} i^3$
- $2^3 + 3^3 + \ldots + n^3 = \sum_{j=1}^{n-1} (j + 1)^3$
- $2^3 + 3^3 + \ldots + n^3 = \sum_{k=0}^{n-2} (k + 2)^3$
Quiz Question 1: Which of the options below are correct sigma notation for the sum:

\[3 + 5 + 7 + 9 + 11 + 13 + 15 + 17\]

(a) \[\sum_{i=1}^{8} (3 + 2i)\]

(b) \[\sum_{j=1}^{8} 2(1 + j)\]

(c) \[\sum_{k=-2}^{5} [2 + (k + 3)]\]

(d) \[\sum_{m=0}^{7} (3 + 2m)\]
Theorem A.2:

Let $c$ be any constant. Then

(a) $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$

(b) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$
Theorem A.2:
Let $c$ be any constant. Then

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(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Proof: (a) follows from the distributive law:

$$ca_m + ca_{m+1} + \ldots + ca_n = c(a_m + a_{m+1} + \ldots a_n).$$

(b) follows from the commutativity and associativity of addition.
(c) combine the last two results.
Example:

Find \( \sum_{i=1}^{n} 1 \).

Solution: \( \sum_{i=1}^{n} = 1 + 1 + \ldots + 1 \) \( n \) terms

Example: Prove the formula for the sum of the first \( n \) positive integers:

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]
Telescoping sums:

Consider \( \sum_{j=0}^{n} \left( \frac{1}{j + 1} - \frac{1}{j + 2} \right) \).

This is equal to

\[
\left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots
\]

\[
\ldots + \left( \frac{1}{(n - 1) + 1} - \frac{1}{(n - 1) + 2} \right)
\]

\[
+ \left( \frac{1}{n + 1} - \frac{1}{n + 2} \right) = 1 - \frac{1}{n + 2}
\]
Example:

Prove the formula for the sum of the squares of the first $n$ positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Proof: Let $S = \sum_{i=1}^{n} i^2$. Consider the telescoping sum:

$$\sum_{i=1}^{n} [(1 + i)^3 - i^3] = \ldots$$
Theorem A.3: Let $c$ be a constant and $n$ a positive integer. Then

(a) $\sum_{i=1}^{n} 1 = n$

(b) $\sum_{i=1}^{n} c = n\,c$

(c) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2$
Quiz Question 2: Which of the options below are correct sigma notation for the sum:

\[ 2 + 7 + 12 + 17 + 22 \]

(a) \[ \sum_{i=1}^{n} (2 + 5i) \]

(b) \[ \sum_{j=1}^{5} [12 + 5(j - 2)] \]

(c) \[ \sum_{k=1}^{5} (2 + 5m) \]

(d) \[ \sum_{\ell=0}^{4} [7 + 5(\ell - 1)] \]
Examples:

Evaluate

\[ \sum_{i=1}^{n} i(4i^2 - 3) \]

Find

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left[ \left(\frac{i}{n}\right)^2 + 1 \right] \]
An important example of sigma notation:

\[ 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]

\[ = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

Therefore

\[ e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots = 2.718 \ldots \]
Exercises:

Write in expanded form:

\[ \sum_{j=n}^{n+3} j^2 = n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 \]

Write in sigma notation:

\[ \sum_{i=0}^{6} (-1)^i (3^i) \quad \text{OR} \quad \sum_{k=1}^{7} (-1)^{k+1} (3^{k-1}) \quad \text{OR} \ldots \]